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MAGNETIC FIELD GENERATION DURING SHEAR MOTION OF CONDUCTING LAYERS

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This paper considers the generation of a longitudinal magnetic field between rigid conducting layers of different electrical conductivity which are in shear motion at constant velocity across the flux lines of the initial field. The amplification limits for the generated field due to the finite thickness of the conducting layers are established. The evolution of the magnetic field under various conditions of generation was described using the models of two layers of finite thickness, a semi-infinite layer and a layer of finite thickness, and two semi-infinite layers.

Formulation of the Problem. The shear motion of conductors across the flux lines of the initial magnetic field generated in them induces electric currents in the conductors and leads to generation of a magnetic field whose flux lines are oriented along the direction of motion [1]. Under certain conditions, such motion of conductors produces considerable amplification of the initial field, which can be used to generate strong magnetic fields [2].

Bichenkov [2] considered the generation of a magnetic field in a clearance between rigid (undeformed) conducting half-spaces of identical electrical conductivities which are in shear motion. In the present work, the results obtained in [2] are extended to the case of shear motion of rigid conducting layers which have finite thicknesses and different electrical conductivities.

The problem is solved in the following formulation (Fig. 1). Two absolutely rigid layers of conducting materials with specific electrical conductivities σ_1 (right layer) and σ_2 (left layer) are located parallel to one another and separated by a clearance of width a. The thicknesses of the right and left layers are equal to δ_1 and δ_2, respectively. In the materials of the layers and the surrounding space, a homogeneous magnetic field is generated which has induction B_0 and is directed normally to th layers. The layers move relative to one another in the longitudinal direction at velocity v_0.

As a result of motion in the transverse magnetic field B_0, electric currents are generated in the materials of the layers. These current flow perpendicular to the flux lines of the transverse field and the direction of the shear motion (i.e., perpendicular to the plane in Fig. 1), and a longitudinal magnetic field arises. Obviously, by virtue of the charge conservation law, the total inductive currents flowing in the left and right layers should have opposite directions and identical magnitudes (these currents are assumed to be close on each other at infinity). From this (by the Ampere law) it follows that the generated longitudinal field exists only in the materials of the layers and in the clearance between them and it disappears on the surfaces of the layers facing outward (from the clearance). The inductions of the longitudinal magnetic field in the right and left layers are denoted by B_1 and B_2, respectively, and the induction of the clearance field is denoted by B_c. The clearance field B_c is homogeneous, and the values of B_1 and B_2 are varying along the thicknesses of the layers. All these quantities are functions of time t. Under the formulated conditions, the induction of the initial transverse field B_0 does not vary.

Without loss of generality, we chose a coordinate system in which the left layer is at rest and the right layer moves parallel to it in the longitudinal direction at velocity v_0 (Fig. 1). The x axis of the coordinate system is perpendicular to the layers and the y axis is directed along the direction of longitudinal motion. The currents induced in the layers by shear motion interact with the transverse field B_0, as a result of which the layers are acted upon by oppositely directed longitudinal electromagnetic forces, which tend to decrease the velocity of their

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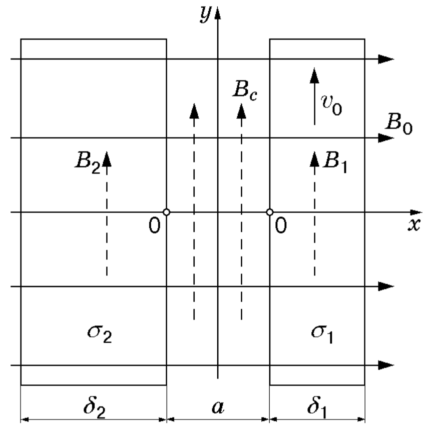

Fig. 1. Diagram of magnetic field generation by shear motion of conducting layers.

shear motion. In the problem considered, the action of these forces is ignored, i.e., it is assumed that they are counterbalanced by some external forces applied to the layers, so that the velocity of shear motion of the layers remains unchanged. The motion of the layers along the x axis is also ruled out.

The evolution of the longitudinal magnetic field in the materials of the right and left layers is described by the following diffusion equations, respectively [1]:

$$\frac{\partial B_1}{\partial t} = \nu_{m1}\,\frac{\partial^2 B_1}{\partial x^2}, \qquad \frac{\partial B_2}{\partial t} = \nu_{m2}\,\frac{\partial^2 B_2}{\partial x^2}. \tag{1}$$

Here $\nu_{m1} = 1/(\mu_0\sigma_1)$ and $\nu_{m2} = 1/(\mu_0\sigma_2)$ are the magnetic viscosities of the materials and $\mu_0 = 4\pi \cdot 10^{-7}$ H/m is the permeability of vacuum. Since the clearance field B_c is homogeneous, the extent of the clearance along the x axis can be considered equal to zero, assuming that the ranges of the x coordinate for the first ($0 < x < \delta_1$) and second ($-\delta_2 < x < 0$) equations of system (1) conjugate with each other at $x = 0$. In this case, the conditions of conjugation of the functions $B_1(x,t)$ and $B_2(x,t)$ at $x = 0$ should apparently be specified by relevant boundary conditions taking into account the actual width of the clearance a between the layers.

The present physical formulation of the problem imply the following boundary conditions. Because the longitudinal magnetic field should be absent in the space outside the layers, it is necessary that it disappear on their outer surfaces at $x = \delta_1$ and $x = -\delta_2$:

$$B_1(\delta_1, t) = 0, \qquad B_2(-\delta_2, t) = 0. \tag{2}$$

In addition, for finite conductivity of the materials of the layers ($\nu_{m1} \neq 0$ and $\nu_{m2} \neq 0$), the passage of surface currents is ruled out, and, therefore, the longitudinal field should be continuous on the layer surfaces facing the clearance. From this, we have one boundary condition at $x = 0$:

$$B_1(0,t) = B_2(0,t) = B_c(t). \tag{3}$$

The other boundary condition at $x = 0$ can be obtained, as in [2], from the Faraday law of electromagnetic induction and differential Ohm's law. The motion of the right layer in the transverse magnetic field B_0 (Fig. 1) and the change of the longitudinal field in the clearance B_c should induce electric currents on the layer surfaces facing each other. The volume densities of these currents are $j_1 = (\partial B_1/\partial x)/\mu_0$ and $j_2 = (\partial B_2/\partial x)/\mu_0$. Using Ohm's law to relate these currents with the induction e.m.f. in the circuit enclosing the clearance (with allowance for the directions of the currents on the surfaces of the right and left layers), we arrive at one more boundary condition at $x = 0$:

$$a\,\frac{dB_c}{dt} = B_0 v_0 + \nu_{m1}\,\frac{\partial B_1}{\partial x} - \nu_{m2}\,\frac{\partial B_2}{\partial x}. \tag{4}$$

The initial conditions for the generated field are zero: $B_2(x,0) = B_1(x,0) = B_c(0) = 0$.

The inductive currents generated in the layers during shear motion are first concentrated near their inner surfaces and then, as a result of diffusion, they penetrate into the depth of the layer materials. Accordingly, the longitudinal field generated by these currents in the clearance gradually diffuses from the clearance to the layer material. Apparently, if the time of motion of the layers is much less than the time of diffusion of the field over their thicknesses, the final cross-sectional dimensions of the layers do not influence magnetic field generation. Hence, the extent of the layers along the x axis can be considered infinite. In view of this circumstance, three possible versions of the problem are considered: field generation between two layers of finite thickness, between a semi-infinite layer and a layer of finite thickness, and between two semi-infinite layers.

Layers of Finite Thickness. Introducing the dimensionless coordinate $x' = x/\sqrt{\delta_1\delta_2}$ and the time $t' = t\sqrt{\nu_{m1}\nu_{m2}}/(\delta_1\delta_2)$, we write system (1) as

$$\frac{\partial B_1'}{\partial t'} = \gamma\,\frac{\partial^2 B_1'}{\partial (x')^2}, \qquad \frac{\partial B_2'}{\partial t'} = \frac{1}{\gamma}\,\frac{\partial^2 B_2'}{\partial (x')^2}, \tag{5}$$

where the scale of induction of the generated field is the induction of the initial transverse field B_0 and the dimensionless parameter γ characterizes the ratio of the magnetic viscosities of the layers: $\gamma = \sqrt{\nu_{m1}/\nu_{m2}}$.

Boundary conditions (2)–(4) become

$$x' = æ\colon\quad B_1'(æ, t') = 0, \qquad x' = -1/æ\colon\quad B_2'(-1/æ, t') = 0,$$

$$x' = 0\colon\quad B_2'(0, t') = B_1'(0, t') = B_c'(t'), \tag{6}$$

$$a'\,\frac{dB_c'}{dt'} = \mathrm{Re}_m + \gamma\,\frac{\partial B_1'}{\partial x'} - \frac{1}{\gamma}\,\frac{\partial B_2'}{\partial x'},$$

where $a' = a/\sqrt{\delta_1\delta_2}$ is the dimensionless width of the clearance between the layers, the dimensionless parameter $æ = \sqrt{\delta_1/\delta_2}$ characterizes the ratio of the layer thicknesses, and the magnetic Reynolds number $\mathrm{Re}_m = v_0\sqrt{\delta_1\delta_2/(\nu_{m1}\nu_{m2})}$ describes the relationship between the rate of field generation (characterized by the velocity of shear motion v_0) and the rate of field diffusion through the layer materials.

Applying an integral Laplace transform [3] in time to system (5), for the images of the functions of the generated field

$$B_1^*(x', p) = \int\limits_0^\infty B_1'(x', t')\exp(-pt')\,dt', \quad B_2^*(x', p) = \int\limits_0^\infty B_2'(x', t')\exp(-pt')\,dt',$$

we obtain the ordinary linear differential equations of the second order

$$\gamma\,\frac{d^2 B_1^*}{d(x')^2} - pB_1^* = 0, \qquad \frac{1}{\gamma}\,\frac{d^2 B_2^*}{d(x')^2} - pB_2^* = 0, \tag{7}$$

whose general solution is written as

$$B_1^* = A_1\sinh\,(\sqrt{p/\gamma}\,x') + A_2\cosh\,(\sqrt{p/\gamma}\,x'), \qquad B_2^* = A_3\sinh\,(\sqrt{\gamma p}\,x') + A_4\cosh\,(\sqrt{\gamma p}\,x').$$

The constants of integration A_1, A_2, A_3, and A_4 included in this solution are determined from the boundary conditions of the problem (6), to which the Laplace transformation is also applicable. As a result, the images of the required functions are written as

$$B_1^* = \frac{\mathrm{Re}_m}{pZ(p)}\,\sinh\Big(\frac{\sqrt{\gamma}}{æ}\,\sqrt{p}\Big)\sinh\Big(\frac{æ}{\sqrt{\gamma}}\,\sqrt{p}\Big(1 - \frac{x'}{æ}\Big)\Big),$$

$$B_2^* = \frac{\mathrm{Re}_m}{pZ(p)}\sinh\Big(\frac{\sqrt{\gamma}}{æ}\,\sqrt{p}(1 + æx')\Big)\sinh\Big(\frac{æ}{\sqrt{\gamma}}\,\sqrt{p}\Big),$$

$$B_c^* = \frac{\mathrm{Re}_m}{pZ(p)}\sinh\Big(\frac{\sqrt{\gamma}}{æ}\,\sqrt{p}\Big)\sinh\Big(\frac{æ}{\sqrt{\gamma}}\,\sqrt{p}\Big),$$

where

$$Z(p) = a'p\sinh\Big(\frac{æ}{\sqrt{\gamma}}\,\sqrt{p}\Big)\sinh\Big(\frac{\sqrt{\gamma}}{æ}\,\sqrt{p}\Big) + \sqrt{\gamma p}\,\cosh\Big(\frac{æ}{\sqrt{\gamma}}\,\sqrt{p}\Big)\sinh\Big(\frac{\sqrt{\gamma}}{æ}\,\sqrt{p}\Big) + \sqrt{\frac{p}{\gamma}}\sinh\Big(\frac{æ}{\sqrt{\gamma}}\,\sqrt{p}\Big)\cosh\Big(\frac{\sqrt{\gamma}}{æ}\,\sqrt{p}\Big).$$

The obtained images are single-valued functions of the parameter p. As shown by analysis of solutions of the equation $Z(p) = 0$ considered in a complex region, the only singular points of these images are an infinite number of poles of the first order [3], one of which is located at the point $p = 0$, and the remaining lie on the negative part

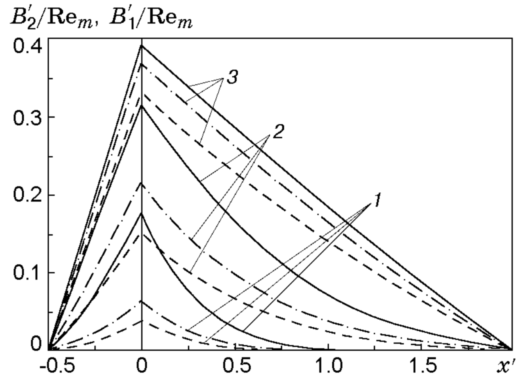

Fig. 2. Evolution of the longitudinal magnetic field in layers of finite thickness ($\gamma = 1$ and $æ = 2$) for $t' = 0.1$ (1), 0.5 (2), and 2 (3) and $a' = 0$ (solid curves), 1 (dot-and-dashed curves), and 2 (dashed curves).

of the real axis. Converting to preimages by summation of the residues of the corresponding images over all these poles [3], we obtain the following expressions for the generated field functions:

$$B'_1(x',t') = \mathrm{Re}_m \left[\frac{æ\gamma(1 - x'/æ)}{æ^2+\gamma^2} + \sum_{k=1}^{\infty} \frac{\exp(-q_k^2 t')}{q_k^2 Z_1(q_k)} \sin\Big(\frac{æ}{\sqrt{\gamma}}\, q_k\Big(1 - \frac{x'}{æ}\Big)\Big) \sin\Big(\frac{\sqrt{\gamma}}{æ}\, q_k\Big)\right],$$

$$B'_2(x',t') = \mathrm{Re}_m \left[\frac{æ\gamma(1 + æx')}{æ^2+\gamma^2} + \sum_{k=1}^{\infty} \frac{\exp(-q_k^2 t')}{q_k^2 Z_1(q_k)} \sin\Big(\frac{æ}{\sqrt{\gamma}}\, q_k\Big) \sin\Big(\frac{\sqrt{\gamma}}{æ}\, q_k(1 + æx')\Big)\right], \qquad (8)$$

$$B'_c(t') = \mathrm{Re}_m \left[\frac{æ\gamma}{æ^2+\gamma^2} + \sum_{k=1}^{\infty} \frac{\exp(-q_k^2 t')}{q_k^2 Z_1(q_k)} \sin\Big(\frac{æ}{\sqrt{\gamma}}\, q_k\Big) \sin\Big(\frac{\sqrt{\gamma}}{æ}\, q_k\Big)\right].$$

Here

$$Z_1(q_k) = \frac{1}{2}\Big(\frac{\gamma}{æ} + \frac{æ}{\gamma}\Big) \cos\Big(\frac{æ}{\sqrt{\gamma}}\, q_k\Big) \cos\Big(\frac{\sqrt{\gamma}}{æ}\, q_k\Big) - \Big(a' + \frac{1}{2}\Big(æ + \frac{1}{æ}\Big)\Big) \sin\Big(\frac{æ}{\sqrt{\gamma}}\, q_k\Big) \sin\Big(\frac{\sqrt{\gamma}}{æ}\, q_k\Big)$$

$$+ \frac{1}{2\sqrt{\gamma}\, q_k}\Big(1 - a'\, \frac{\gamma}{æ}\, q_k^2\Big) \sin\Big(\frac{æ}{\sqrt{\gamma}}\, q_k\Big) \cos\Big(\frac{\sqrt{\gamma}}{æ}\, q_k\Big) + \frac{\sqrt{\gamma}}{2q_k}\Big(1 - a'\, \frac{æ}{\gamma}\, q_k^2\Big) \cos\Big(\frac{æ}{\sqrt{\gamma}}\, q_k\Big) \sin\Big(\frac{\sqrt{\gamma}}{æ}\, q_k\Big),$$

where q_k are positive solutions of the equation

$$a' q \sin\Big(\frac{æ}{\sqrt{\gamma}}\, q\Big) \sin\Big(\frac{\sqrt{\gamma}}{æ}\, q\Big) - \sqrt{\gamma} \cos\Big(\frac{æ}{\sqrt{\gamma}}\, q\Big) \sin\Big(\frac{\sqrt{\gamma}}{æ}\, q\Big) - \frac{1}{\sqrt{\gamma}} \sin\Big(\frac{æ}{\sqrt{\gamma}}\, q\Big) \cos\Big(\frac{\sqrt{\gamma}}{æ}\, q\Big) = 0,$$

which is obtained from the equation $Z(p) = 0$ by the replacement $p = -q^2$.

It should be noted that the series in relations (8) converge rapidly, so that in the present calculations, we can restrict ourselves to only a few first terms. According to the solutions obtained, in magnetic field generation by shear motion of layers of finite thickness there is a limiting level of induction of the generated field, to which the induction tends monotonically as $t' \to \infty$. The limiting intensity of the clearance field does not depend on clearance width and is

$$B'_{c,\mathrm{lim}} = \mathrm{Re}_m/(æ/\gamma + \gamma/æ) \qquad (9)$$

or, after conversion to dimensional quantities,

$$B_{c,\mathrm{lim}} = B_0 v_0/(\nu_{m1}/\delta_1 + \nu_{m2}/\delta_2).$$

According to (9), for the same value of Re_m, the maximum limiting amplification of the clearance field is reached for $æ = \gamma$ (i.e., for $\delta_1\nu_{m2} = \delta_2\nu_{m1}$). In this case, $B'_{c,\mathrm{lim}} = 0.5\,\mathrm{Re}_m$.

At the limit (as $t' \to \infty$), the distributions of the induction of the generated field over the thicknesses of the layers tend to linear ones (Fig. 2):

$$B'_1 \approx \mathrm{Re}_m(1 - x'/æ)/(æ/\gamma + \gamma/æ), \qquad B'_2 \approx \mathrm{Re}_m(1 + æx')/(æ/\gamma + \gamma/æ),$$

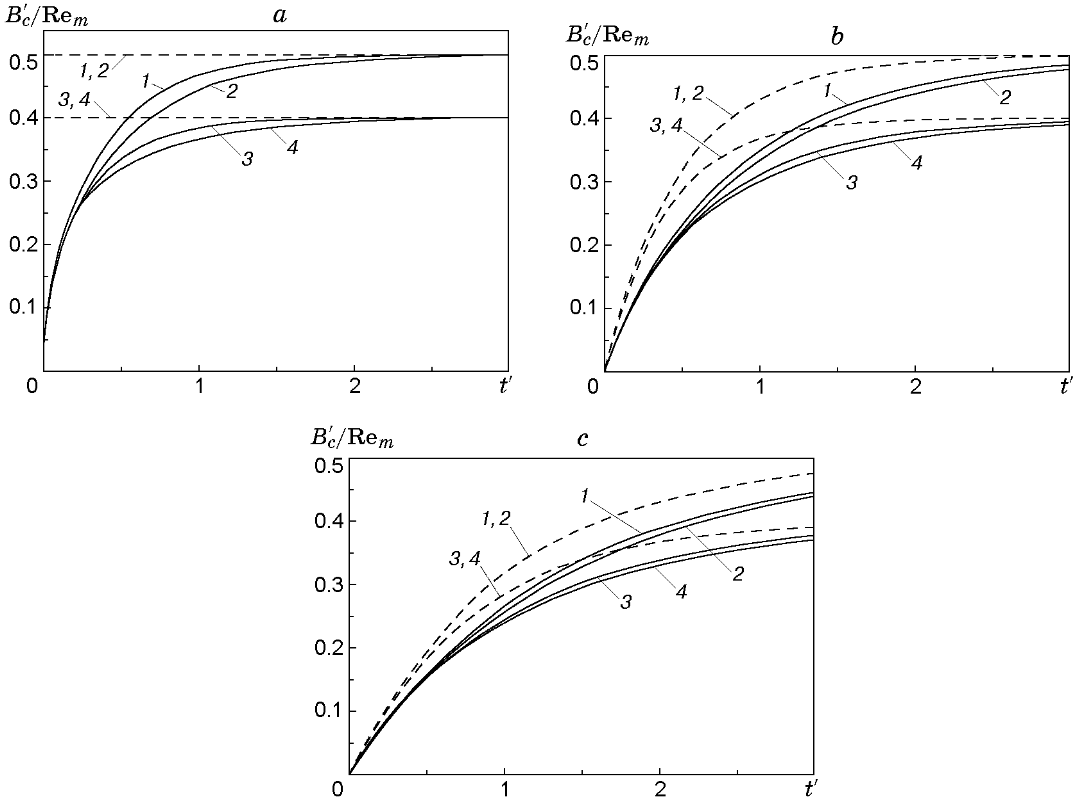

Fig. 3. Change of the longitudinal magnetic field between conducting layers of finite thickness for $a' = 0$ (a), 1 (b), and 2 (c): curves 1 refer to $\gamma = 1$ and $æ = 1$, curves 2 to $\gamma = 2$ and $æ = 2$, curves 3 to $\gamma = 1$ and $æ = 2$, and curves 4 to $\gamma = 2$ and $æ = 1$ (solid curves refer to the exact solution and dashed curves refer to the approximation of "electrically thin" layers).

which corresponds to uniform distribution of the inductive-current density in the layers:

$$j_1 = \frac{1}{\mu_0}\,\frac{\partial B_1}{\partial x} \approx -\frac{B_0 v_0}{\mu_0 \delta_1(\nu_{m1}/\delta_1 + \nu_{m2}/\delta_2)}, \qquad j_2 = \frac{1}{\mu_0}\,\frac{\partial B_2}{\partial x} \approx \frac{B_0 v_0}{\mu_0 \delta_2(\nu_{m1}/\delta_1 + \nu_{m2}/\delta_2)}.$$

Thus, from the analysis performed it follows that the evolution of the generated field at the entrance into the limiting level can be describe using the assumption of uniform distribution of inductive currents over the thicknesses of the layers. This assumption (referred to as the approximation of "electrically thin" layers) simplifies the solution of the problem considerably.

Within the framework of the approximation of "electrically thin" layers, the variation of the clearance magnetic field B_c is described by the ordinary linear differential equation

$$a\,\frac{dB_c}{dt} + \Big(\frac{\nu_{m1}}{\delta_1} + \frac{\nu_{m2}}{\delta_2}\Big)B_c = B_0 v_0, \tag{10}$$

which is obtained from boundary condition (4) if in the latter we set $\partial B_1/\partial x = -B_c/\delta_1$ and $\partial B_2/\partial x = B_c/\delta_2$. The solution of Eq. (10) subject to the initial condition $B_c(0) = 0$ is written as

$$B_c(t) = \frac{B_0 v_0}{\nu_{m1}/\delta_1 + \nu_{m2}/\delta_2}\Big\{1 - \exp\Big[-\frac{1}{a}\Big(\frac{\nu_{m1}}{\delta_1} + \frac{\nu_{m2}}{\delta_2}\Big)t\Big]\Big\}. \tag{11}$$

Figure 3 shows the variation of the clearance magnetic field for various values of the parameters a', $æ$, and γ obtained from the exact solution (8) using the approximation of "electrically thin" layers. From Fig. 3 it follows that an increase in the width of the clearance a' leads to a decrease in the rate of attainment of the limiting level of the generated field (9). In this case, the approximate solution (11) yields an overestimated growth rate of the field in the initial stages of generation and in the case of no clearance, the limiting value of the longitudinal magnetic field is established instantaneously.

It should be noted that using the approximation of "electrically thin" layers a result closer to the exact solution is obtained if the Faraday law of electromagnetic induction (10) is written not for a circuit enclosing the clearance but for a circuit of width $a + (\delta_1 + \delta_2)/2$ that passes through the middle of the layers. In this case, relation (11) becomes

$$B_c(t) = \frac{B_0 v_0}{\nu_{m1}/\delta_1 + \nu_{m2}/\delta_2} \Big[1 - \exp\Big(-\frac{\nu_{m1}/\delta_1 + \nu_{m2}/\delta_2}{a + (\delta_1 + \delta_2)/2}\, t\Big)\Big].$$

If the properties of the layers determining the screening of the generated field are markedly different (the times of magnetic-field diffusion δ_1^2/ν_{m1} and δ_2^2/ν_{m2} for the layers differ markedly), the approximation of an "electrically thin" layer can be used for one of the layers (for which the time of diffusion is smaller). If we assume that the right layer (see Fig. 1) is "electrically thin" with constant current density $j_1 = -B_c/(\mu_0\delta_1)$, the problem reduces to solution of the field-diffusion equation only for the left layer [second equation in system (1)] subject to the boundary conditions

$$x = -\delta_2\colon \quad B_2(-\delta_2, t) = 0,$$

$$x = 0\colon \quad a\,\frac{dB_c}{dt} = B_0 v_0 - \frac{\nu_{m1}}{\delta_1}\, B_c - \nu_{m2}\,\frac{\partial B_2}{\partial x},$$

where $B_c(t) = B_2(0, t)$. Using the same dimensionless parameters as before, after applying a Laplace transform, we obtain the following expression for the image of the induction of the longitudinal field in the left layer:

$$B_2^*(x', p) = \mathrm{Re}_m \sinh\Big(\frac{\sqrt{\gamma}}{æ}\sqrt{p}(1 + æx')\Big) \Big/ \Big[p\Big(\Big(a'p + \frac{\gamma}{æ}\Big)\sinh\Big(\frac{\sqrt{\gamma}}{æ}\sqrt{p}\Big) + \sqrt{\frac{p}{\gamma}}\,\cosh\Big(\frac{\sqrt{\gamma}}{æ}\sqrt{p}\Big)\Big)\Big].$$

The set of singular points for this image includes only an infinite number of poles of the first order (at the point $p = 0$ and on the negative part of the real axis). Summing up the residues over all these poles [3], we obtain the preimage of the field:

$$B_2'(x', t') = \mathrm{Re}_m \Big\{\frac{æ\gamma(1 + æx')}{æ^2 + \gamma^2} + \sum_{k=1}^{\infty} \sin\Big(\frac{\sqrt{\gamma}}{æ}\, q_k(1 + æx')\Big)\exp(-q_k^2 t')$$

$$\Big/\Big[q_k^2\Big(\frac{1}{2q_k}\,\frac{\sqrt{\gamma}}{æ}\Big(\frac{\gamma}{æ} + \frac{æ}{\gamma} - a'q_k^2\Big)\cos\Big(\frac{\sqrt{\gamma}}{æ}\, q_k\Big) - \Big(a' + \frac{1}{2æ}\Big)\sin\Big(\frac{\sqrt{\gamma}}{æ}\, q_k\Big)\Big)\Big]\Big\}, \qquad (12)$$

where q_k are positive solutions of the equation

$$\Big(\frac{\gamma}{æ} - a'q^2\Big)\sin\Big(\frac{\sqrt{\gamma}}{æ}\, q\Big) + \frac{q}{\gamma}\,\cos\Big(\frac{\sqrt{\gamma}}{æ}\, q\Big) = 0. \qquad (13)$$

The law of variation of the clearance magnetic field $B_c'(t')$ is obtained from (12) at $x' = 0$. As in the approximation of two "electrically thin" layers, the accuracy of the solution of (12) for early stages of generation can be increased by choosing a different circuit for the Faraday law of electromagnetic induction (boundary conditions at $x = 0$). In this case, a more exact result is obtained using a circuit of width $a + \delta_1/2$ that passes through the surface of the left layer facing the clearance (see Fig. 1) and the middle of the right ("electrically thin") layer. With such a choice, the solution is obtained if in (12) and (13) the quantity a' is replaced by $a' + æ/2$.

As is noted above, to keep the velocity of shear motion v_0 constant, the layers should be acted upon by external forces that balance the electromagnetic forces due to the transverse field B_0. For the right layer, the electromagnetic force is directed downward. The magnitude of this force per unit surface of the layer is evaluated from the formula

$$f_A = \int_0^{\delta_1} j_1 B_0\, dx = \frac{B_0}{\mu_0}\int_0^{\delta_1} \frac{\partial B_1}{\partial x}\, dx = \Big|\frac{B_0}{\mu_0}\,(B_1(\delta_1) - B_1(0))\Big| = \frac{B_0 B_c}{\mu_0}$$

and is completely determined by the current intensity of the clearance field and the induction of the transverse field. Apparently, the same force directed upward acts on the left layer.

We use the obtained results to estimate the limiting amplification of the magnetic field during shear motion of conducting plates that arise in reality because of the finiteness of their thicknesses. For identical plates ($\delta_1 = \delta_2 = \delta$ and $\nu_{m1} = \nu_{m2} = \nu_m$), the limiting field amplification is determined as follows: $B_{c,\lim}/B_0 = 0.5 v_0\delta/\nu_m$. In the case of copper plates 1 cm thick moving at relative velocity equal to several tens of meters per second, $B_{c,\lim}/B_0 \approx 10$. To generate fields of the megagauss level, it is necessary to produce an initial field $B_0 \sim 10$ T.

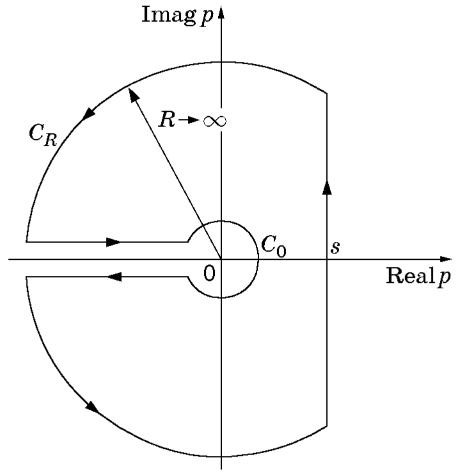

Fig. 4. Oriented circuit in a complex plane for calculation of the Mellin integral.

Semi-Infinite Layer and a Layer of Finite Thickness. We assume that for one of the layers, the time of magnetic-field diffusion is so great (compared to the other layer), that its thickness does not influence field generation. Then, this layer can be considered semi-infinite (for example, the left layer in Fig. 1).

As the spatial scale, we use the thickness of the right (finite) layer δ_1, defining the dimensionless coordinate as $x' = x/\delta_1$ and the dimensionless time as $t' = t\delta_1/\sqrt{\nu_{m1}\nu_{m2}}$. In this case, the form of the field diffusion equations, written in dimensionless variables, retain their form (5). As regards boundary conditions (6), at $x' = 0$, they remain unchanged but the dimensionless width of the clearance a' and the magnetic Reynolds number Re_m are defined differently:

$$a' = a/\delta_1, \qquad \mathrm{Re}_m = v_0\delta_1/\sqrt{\nu_{m1}\nu_{m2}}.$$

The boundary condition on the rear surface of the right layer ($x = \delta_1$) is written in the dimensionless variables as

$$x' = 1, \qquad B_1'(1, t') = 0.$$

The identical condition on the rear surface of the left (semi-infinite) layer is eliminated, being replaced by the requirement of boundedness of the function $B_2'(x', t')$ as $x' \to -\infty$.

The general solutions of Eqs. (7) for the images of the generated field functions (taking into account the condition of boundedness of the solution for B_2^* as $x' \to -\infty$) become

$$B_1^* = A_1 \sinh\left(\sqrt{p/\gamma}\, x'\right) + A_2 \cosh\left(\sqrt{p/\gamma}\, x'\right), \qquad B_2^* = A_3 \exp\left(\sqrt{\gamma p} x'\right).$$

After determining the constants of integration A_1, A_2, and A_3 from the boundary conditions, for the image of the function of the longitudinal field in the clearance, we obtain the expression

$$B_c^*(p) = \mathrm{Re}_m/[p(a'p + \sqrt{\gamma p} \coth\left(\sqrt{p/\gamma}\right) + \sqrt{p/\gamma})]. \tag{14}$$

To invert this image, we use the Mellin formula [3]

$$B_c'(t') = \frac{1}{2\pi i} \int\limits_{s-i\infty}^{s+i\infty} B_c^*(p) \exp(pt')\, dp, \tag{15}$$

where i is imaginary unity and s is chosen so that all singular points of the integrand lie on the left of the straight line Real$(p) = s$ (Fig. 4).

The point $p = 0$ is a branch point [3] of the image (14). Therefore, to distinguish the single-valued branch of the image in evaluating of the integral (15) in the plane of the complex parameter p, we choose an oriented closed circuit of integration with a cut along the negative part of the real axis (Fig. 4). Singular points of the function $B_c^*(p)$ (14) are absent inside this circuit, and, therefore, the integral over this circuit is equal to zero [3]. Written as the sum of integrals over separate segments of the circuit, this integral includes, apart from (15), nonzero contribution from the integrals over the edges of the cut and the integral over a circle C_0 of infinitesimal radius which includes the coordinate origin, at which the first-order pole of the function (14) is located. Writing the integrals over the edges of the cut and taking into account that the integral over the circle C_0 (Fig. 4) is equal to the residue of the integrand at the point $p = 0$,

$$\mathrm{Res}\Big(\frac{\mathrm{Re}_m \exp(pt')}{p(a'p + \sqrt{\gamma p}\coth(\sqrt{p/\gamma}) + \sqrt{p/\gamma})}\Big)_{p=0} = \frac{\mathrm{Re}_m}{\gamma},$$

we obtain the law of variation of the field generated in the clearance:

$$B'_c(t') = \frac{\mathrm{Re}_m}{\gamma}\Bigg(1 - \frac{\sqrt{\gamma}}{\pi}\int\limits_0^\infty \frac{\exp(-qt')\,dq}{\sqrt{q}((a'q - \sqrt{\gamma q}\cot(\sqrt{q/\gamma}))^2 + q/\gamma)}\Bigg). \tag{16}$$

As $t' \to \infty$, the integral in (16) tends to zero, and the ultimate attainable magnitude of the field in the clearance between the semi-infinite layer and the layer of finite thickness is given by the expression $B'_{c,\lim} = \mathrm{Re}_m/\gamma$. In conversion to the dimensional quantities $B_{c,\lim} = B_0 v_0 \delta_1/\nu_{m1}$, this value depends only on the characteristics of the layer of finite thickness.

Further simplification of the solution (16) with attainment of the final relations can be obtained for the case where the layer of finite thickness is considered "electrically thin" (this approximation, as is note above, is valid as $t' \to \infty$). In this case, the image (14) of the clearance field is transformed:

$$B^*_c(p) = \mathrm{Re}_m/[p(a'p + \sqrt{p/\gamma} + \gamma)]. \tag{17}$$

The result of inversion of the image (17) depends on the form of roots of the quadratic equation

$$q^2 + q/(a'\sqrt{\gamma}) + \gamma/a' = 0. \tag{18}$$

For $4a'\gamma^2 < 1$, the given equation has two real roots (both negative)

$$q_1 = -(1 + \sqrt{1 - 4a'\gamma^2})/(2a'\sqrt{\gamma}), \qquad q_2 = -(1 - \sqrt{1 - 4a'\gamma^2})/(2a'\sqrt{\gamma})$$

and the preimage of the clearance field can be written as [4]

$$B'_c(t') = \frac{\mathrm{Re}_m}{\gamma}\Big[1 - \frac{2a'\gamma^2}{\sqrt{1 - 4a'\gamma^2}}\Big(\frac{\exp(q_2^2 t')\,\mathrm{erfc}(|q_2|\sqrt{t'})}{1 - \sqrt{1 - 4a'\gamma^2}} - \frac{\exp(q_1^2 t')\,\mathrm{erfc}(|q_1|\sqrt{t'})}{1 + \sqrt{1 - 4a'\gamma^2}}\Big)\Big],$$

where $\mathrm{erfc}(z) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int\limits_z^\infty \exp(-\xi^2)\,d\xi$ is an additional Gauss error function [5].

For $4a'\gamma^2 = 1$, Eq. (18) has a multiple root $q_{1,2} = -1/(2a'\sqrt{\gamma})$ and the preimage for (17) is written as [4]

$$B'_c(t') = \frac{\mathrm{Re}_m}{\gamma}\Big[1 - \Big(1 - 2\frac{\gamma}{a'}t'\Big)\exp\Big(\frac{\gamma}{a'}t'\Big)\,\mathrm{erfc}\Big(\sqrt{\frac{\gamma}{a'}t'}\Big) - \frac{2}{\sqrt{\pi}}\sqrt{\frac{\gamma}{a'}t'}\Big].$$

For $4a'\gamma^2 > 1$, the solutions of Eq. (18) are complex conjugate, and the variation of the clearance field is described by the relation [4]

$$B'_c(t') = \frac{\mathrm{Re}_m}{\gamma}\Big[1 - \mathrm{Real}\Big(w\Big(\frac{\sqrt{4a'\gamma^2-1}+i}{2a'\sqrt{\gamma}}\sqrt{t'}\Big)\Big) - \frac{1}{\sqrt{4a'\gamma^2-1}}\,\mathrm{Imag}\Big(w\Big(\frac{\sqrt{4a'\gamma^2-1}+i}{2a'\sqrt{\gamma}}\sqrt{t'}\Big)\Big)\Big],$$

where $w(z) = \exp(-z^2)\Big(1 + \dfrac{2i}{\sqrt{\pi}}\displaystyle\int\limits_0^z \exp(\xi^2)\,d\xi\Big)$ is the probability integral with a complex argument [5].

Under the approximation of an "electrically thin" layer for the early stages of generation, the calculation accuracy for the given relations [beginning form relation (17)] can be increased by replacing the quantity a' by $a' + 1/2$, which, in writing boundary conditions (4), corresponds to an electric circuit that passes through the middle of an "electrically thin" layer.

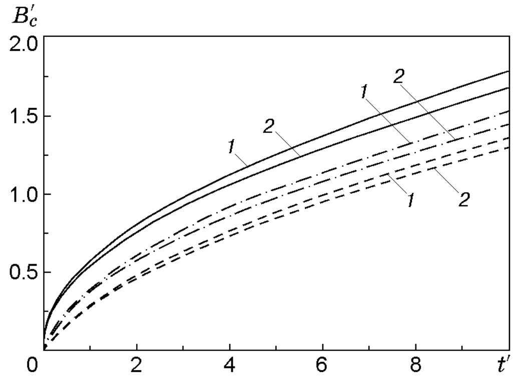

Fig. 5. Variation of the longitudinal magnetic field between conducting semi-infinite layers for $\gamma = 1$ (curves 1) and 2 (curves 2): the solid, dot-and-dashed, and dashed curves refer to $a' = 0$, 1, and 2, respectively.

Semi-Infinite Layers. For the evolution of the generated field in the early stages of generation at times much smaller than the diffusion time for any of the layers, the thicknesses of both layers can be considered infinite. In this case, the dimensionless time t' and the coordinate x' are defined as $t' = tv_0^2/\sqrt{\nu_{m1}\nu_{m2}}$ and $x' = xv_0/\sqrt{\nu_{m1}\nu_{m2}}$, respectively. This makes it possible to retain the previous form of the field-diffusion equations for the layers in the dimensionless variable (5) with the same dimensionless parameter γ as in (5). The boundary conditions for semi-infinite layers are formulated only for $x' = 0$. In the dimensionless variables, they are written as

$$B'_2(0, t') = B'_1(0, t') = B'_c(t'), \qquad a' \frac{dB'_c}{dt'} = 1 + \gamma \frac{\partial B'_1}{\partial x'}\Big|_{x'=0} - \frac{1}{\gamma}\frac{\partial B'_2}{\partial x'}\Big|_{x'=0},$$

where the dimensionless width of the clearance is defined as $a' = av_0/\sqrt{\nu_{m1}\nu_{m2}}$. In addition, the functions $B'_1(x', t')$ and $B'_2(x', t')$ are subjected to the boundedness conditions as $x' \to \infty$ and $x' \to -\infty$, respectively.

The solutions of Eqs. (7) for the images of the required functions satisfying the boundedness conditions at infinity and the boundary conditions at $x' = 0$ are written as

$$B_1^* = \frac{\exp(-\sqrt{p/\gamma}\, x')}{p(a'p + (\sqrt{\gamma} + 1/\sqrt{\gamma})\sqrt{p})}, \qquad B_2^* = \frac{\exp(\sqrt{\gamma p}x')}{p(a'p + (\sqrt{\gamma} + 1/\sqrt{\gamma})\sqrt{p})}. \tag{19}$$

Inversion of the images [4] gives the laws of variation of the longitudinal field in the materials of the right and left layers:

$$\begin{aligned} B'_1(x', t') &= \frac{\sqrt{\gamma}}{\gamma+1}\Big[2\sqrt{\frac{t'}{\pi}}\exp\Big(-\frac{(x')^2}{4\gamma t'}\Big) - \Big(\frac{a'\sqrt{\gamma}}{\gamma+1} + \frac{x'}{\sqrt{\gamma}}\Big)\operatorname{erfc}\Big(\frac{x'}{2\sqrt{\gamma t'}}\Big) \\ &+ \frac{a'\sqrt{\gamma}}{\gamma+1}\exp\Big(\frac{(\gamma+1)^2 t'}{(a')^2\gamma} + \frac{(\gamma+1)x'}{a'\gamma}\Big)\operatorname{erfc}\Big(\frac{(\gamma+1)\sqrt{t'}}{a'\sqrt{\gamma}} + \frac{x'}{2\sqrt{\gamma t'}}\Big)\Big], \quad x' \geqslant 0, \\ B'_2(x', t') &= \frac{\sqrt{\gamma}}{\gamma+1}\Big[2\sqrt{\frac{t'}{\pi}}\exp\Big(-\frac{(x')^2\gamma}{4t'}\Big) - \Big(\frac{a'\sqrt{\gamma}}{\gamma+1} - x'\sqrt{\gamma}\Big)\operatorname{erfc}\Big(-\frac{x'\sqrt{\gamma}}{2\sqrt{t'}}\Big) \\ &+ \frac{a'\sqrt{\gamma}}{\gamma+1}\exp\Big(\frac{(\gamma+1)^2 t'}{(a')^2\gamma} - \frac{(\gamma+1)x'}{a'}\Big)\operatorname{erfc}\Big(\frac{(\gamma+1)\sqrt{t'}}{a'\sqrt{\gamma}} - \frac{x'\sqrt{\gamma}}{2\sqrt{t'}}\Big)\Big], \quad x' \leqslant 0. \end{aligned} \tag{20}$$

Setting $a' = 0$ in the images (19) nd returning to the preimages, we determine how the longitudinal field in the layers varies in the absence of a clearance between them:

$$\begin{aligned} B'_1(x', t') &= \frac{\sqrt{\gamma}}{\gamma+1}\Big[2\sqrt{\frac{t'}{\pi}}\exp\Big(-\frac{(x')^2}{4\gamma t'}\Big) - \frac{x'}{\sqrt{\gamma}}\operatorname{erfc}\Big(\frac{x'}{2\sqrt{\gamma t'}}\Big)\Big], \quad x' \geqslant 0, \\ B'_2(x', t') &= \frac{\sqrt{\gamma}}{\gamma+1}\Big[2\sqrt{\frac{t'}{\pi}}\exp\Big(-\frac{(x')^2\gamma}{4t'}\Big) + x'\sqrt{\gamma}\operatorname{erfc}\Big(-\frac{x'\sqrt{\gamma}}{2\sqrt{t'}}\Big)\Big], \quad x' \leqslant 0. \end{aligned} \tag{21}$$

The law of field generation in a clearance between semi-infinite layers is obtained if any of relations (20) is considered at $x' = 0$. This law is written as

$$B_c'(t') = \frac{a'\gamma}{(\gamma+1)^2}\Big[\exp\Big(\frac{(\gamma+1)^2}{(a')^2\gamma}\,t'\Big)\operatorname{erfc}\Big(\frac{\gamma+1}{a'\sqrt{\gamma}}\,\sqrt{t'}\Big) + \frac{2}{\sqrt{\pi}}\,\frac{\gamma+1}{a'\sqrt{\gamma}}\,\sqrt{t'} - 1\Big]. \qquad (22)$$

The solution for the clearance field (22) remains unchanged when γ is replaced by $1/\gamma$ (it is symmetric about the electrical conductivity of the layers): the change of places of the left and right layers should apparently not influence the generation of the clearance field.

According to (20)–(22), the magnetic field generated by shear motion of semi-infinite layers increases without bound [2]. The dynamics of field generation in the clearance between the layers is shown in Fig. 5 for various values of a' and γ. An increase of the clearance between the layers leads to a decrease in the growth rate of the field in the initial stages of generation. The most rapid increase of the field occurs when the layers slide relative to one another in the absence of a clearance. In this case, as follows from (21) for $x' = 0$, on the boundary of contact of the layers, the generated field varies under the law

$$B_c'(t') = (2/\sqrt{\pi})\sqrt{\gamma t'}/(\gamma+1).$$

It is easy to establish that the above relation is the main term of the asymptotic form of (22) as $t' \to \infty$. From this solution it follows that the presence of a clearance between semi-infinite layers influences the evolution of the magnetic field generated by shear motion only on a limited time interval. As $t' \to \infty$, the rate of field generation does not depend on the magnitude of the clearance. As regards the effect of the parameter γ, for constant value of the product of the magnetic viscosities (which determines the chosen time scale), the most rapid increase in the field between the layers is attained for layers having identical electrical conductivity $\gamma = 1$ (Fig. 5).

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